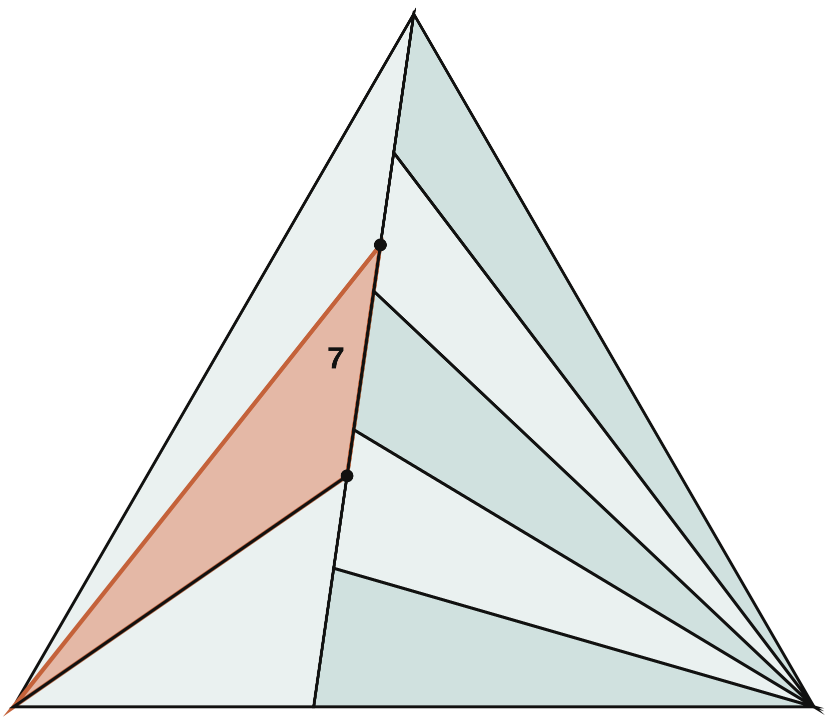
Eight Fair *Slices*

Name ____________________________ Date ______________

Suggested grades 8–12 · 20–40 minutes · triangle area, ratios, the Pythagorean theorem

A bakery's triangular sheet cake — a perfect equilateral triangle — must feed eight people, and every one of them will be watching the knife. The baker makes one long cut from the top corner down to the base, then fans three straight cuts from the left corner and five from the right corner out to points along that first cut. The eight triangular pieces look wildly unequal — yet every piece contains exactly the same amount of cake.

> **The puzzle.** The figure shows the cuts. All eight triangles have **equal area**, and the marked stretch of the long cut — between two of the fan points — has length **7**. **How long is each side of the big triangle?**



Everything you need is in the picture — the equal areas are the machine that determines every length.

SHOW YOUR WORK

Hints — read one at a time

Fold this page away until you need it. After each hint, go back and try again before reading the next.

HINT 1 — THE ONE TOOL

Triangles that share the same apex, with their bases lying along one straight line, have areas in proportion to their bases — because their heights are identical. This single fact, used twice, unlocks the whole figure.

HINT 2 — COUNT THE SLICES

Use the tool on the two big triangles flanking the long cut. You can find their area ratio without measuring anything: just count how many of the eight equal slices sit on each side. That ratio tells you exactly where the long cut meets the base.

HINT 3 — THE FANS, THEN PYTHAGORAS

Use the very same tool again inside one fan: those slices share an apex too, and their bases all lie along the long cut. Equal areas means the marked 7 is one of several identical stretches — so you know the whole cut's length. To finish, drop the altitude from the top corner (in an equilateral triangle it lands on the base's midpoint) and let the Pythagorean theorem relate the cut's length to the side. No trigonometry needed.

KEEP WORKING

Solution

Each side of the triangle is 24. Here is the chase, one idea at a time.

The one tool

Two triangles that share the same apex, with bases along one straight line, have areas in proportion to their bases — their heights (the distance from the shared apex to that line) are identical. Used twice, this fact unlocks everything.

Step 1 — count the slices to find the base split

The long cut runs from the top corner C to a point M on the base AB. To its left sit **3** of the equal slices; to its right sit **5**. So the two flanking triangles ACM and CMB have areas in the ratio 3 : 5. They share the apex C, and their bases AM and MB lie along the same line — so by the tool, **AM : MB = 3 : 5**. The point M sits exactly 3/8 of the way along the bottom. The count of pieces was not decoration; it was the first measurement.

Step 2 — the fans slice the long cut evenly

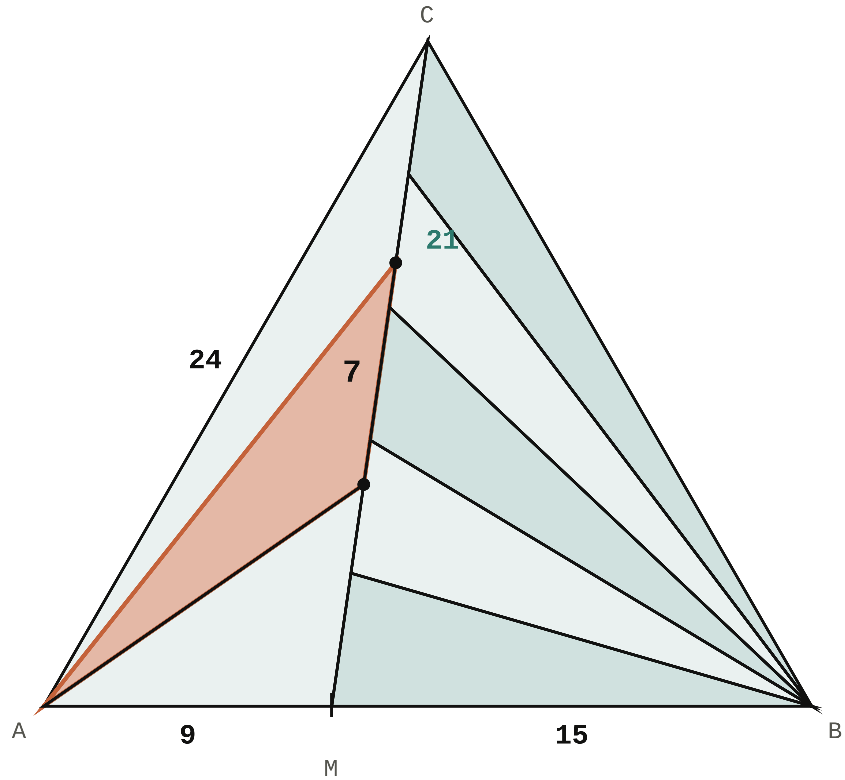
The three left-hand slices share the apex A, and their bases are stretches of the cut CM — bases on one common line again. Equal areas force **equal bases**: the left fan cuts CM into three identical parts (and the right fan, sharing apex B, cuts the same segment into five identical parts). The marked stretch is one of the left fan's thirds, so **CM = 3 × 7 = 21**.

Step 3 — the Pythagorean theorem finishes it

Let s be the side of the triangle, and drop the altitude from C. In an equilateral triangle it lands on the exact midpoint of the base, s/2 from either end — while M sits at 3s/8. So the altitude's foot and M are only s/2 – 3s/8 = **s/8** apart. The altitude's squared length is the familiar $3s^2/4$. The cut CM is the hypotenuse of a right triangle with legs s/8 and the altitude:

$$\mathbf{CM^2 = (s/8)^2 + 3s^2/4 = s^2/64 + 48s^2/64 = 49s^2/64, \text{ so } CM = 7s/8}$$

A ratio of exactly seven-eighths, from a figure built out of 3s and 5s. Setting 7s/8 = 21 gives **s = 24**. The solved figure on the next page fills in every length.



Checking the answer

A side-24 equilateral triangle has area 144 times the square root of 3, about 249.4 square units, so each slice should hold 18 times the square root of 3, about 31.2. The rust slice has base 7 along the cut, and its apex A sits the right distance away for exactly that area. Every slice checks out.

The hidden pattern (worth showing the class)

The clean seven-eighths was not luck. The two triangles flanking the cut are ACM, with sides 24, 9, 21 = 3 × (8, 3, 7), and CMB, with sides 21, 15, 24 = 3 × (7, 5, 8). These are the two smallest **integer triangles containing a perfect 60° angle** — the 60° cousins of the Pythagorean 3-4-5: in a 60° triangle the rule is $a^2 + b^2 - ab = c^2$, and $3^2 + 8^2 - 24 = 49 = 5^2 + 8^2 - 40$. Glue those two famous triangles along their shared side of 7 and they reassemble into an equilateral triangle of side 8 — this figure is exactly that gluing, scaled by three. The puzzle was rigged by number theory.

Extension challenges — for early finishers or discussion

1. Our fan trick cuts a triangle into any number of equal-area triangular pieces. Ask students to cut a **square** into triangles of equal area. Even numbers come easily — but three, five, or seven pieces will defeat everyone, forever: **Monsky's theorem** (1970) proves a square cannot be divided into an odd number of equal-area triangles, and the only known proof uses 2-adic number theory.

2. For a regular pentagon and beyond, **Kasimatis's theorem** (1989) says fair triangle slicing works only when the piece count is a multiple of the number of sides. A pentagon accepts 5, 10, 15 — and nothing else. Students can explore both walls with the free interactive fair-slice machine at **recreatemath.com/supplemental/eight-fair-slices.php**.

References & further reading

This puzzle was inspired by “Great Lengths,” a rectangle-dissection puzzle in the July/August 2026 issue of Scientific American; their math-puzzle series lives at www.scientificamerican.com/games/math-puzzles. Translating it into triangles leads into a deep corner of geometry called **equidissection** — cutting shapes into triangles of equal area.

Monsky’s theorem. A square cannot be cut into an odd number of equal-area triangles. Posed by Fred Richman in the American Mathematical Monthly in the 1960s, partially solved by John Thomas, and settled by Paul Monsky in “On Dividing a Square into Triangles” (Amer. Math. Monthly 77, 1970, 161–164). en.wikipedia.org/wiki/Monsky%27s_theorem

Elaine Kasimatis, “Dissections of Regular Polygons into Triangles of Equal Areas” (Discrete & Computational Geometry 4, 1989, 375–381) — for a regular polygon with five or more sides, an equal-area triangle dissection into m pieces exists if and only if m is a multiple of the number of sides.

Equidissection — the wider landscape, including the “spectrum” of a polygon, a concept introduced by Kasimatis and Sherman Stein. Stein’s survey “Cutting a Polygon into Triangles of Equal Areas” (Mathematical Intelligencer 26, 2004) is a friendly tour; his book with Sandor Szabo, Algebra and Tiling, covers the proofs. en.wikipedia.org/wiki/Equidissection

Integer triangles with a 60° angle — the Eisenstein-triple cousins of Pythagorean triples; (3, 5, 7) is the smallest, and it is exactly the arithmetic hiding inside this puzzle’s 3 : 5 split and length-7 mark. en.wikipedia.org/wiki/Integer_triangle

Peter Winkler, Mathematical Puzzles: A Connoisseur’s Collection — includes the classic fair-division cousin of our machine: cutting a square cake with icing on the sides into pieces with equal cake and equal icing.

The interactive supplement. A fair-slice machine that cuts any regular polygon (3–12 sides) into equal-area triangles, verifies every piece live, and reveals the forbidden counts — plus a step-by-step animation of this puzzle’s solution: recreatemath.com/supplemental/eight-fair-slices.php